Revisit Optimal PMU Placement with Full Zero-Injection Cluster and Redundancy Sharing

Hongxing Ye, *Senior Member, IEEE*, Chuyue Tian, Yinyin Ge, and Lei Wu, *Fellow, IEEE*

***Abstract*—Due to the expensive deployment cost, finding the minimum Phasor Measurement Unit (PMU) to meet the observability requirement has been attractive since the invention of PMU. Many research efforts have been put to zero-injection nodes (ZINs) in optimal PMU placement (OPP). We revisit OPP and define a full Zero-injection cluster (ZIC) as a set of nodes adjacent to ZINs together with at least one ZIN, establishing a novel ZIC-based OPP model. Compared with the existing literature, the full ZIC can cover more nodes, resulting in a better solution. To find all full ZIC, we propose a dynamic adjacency matrix based full-ZIC search algorithm, which has quadratic-time complexity. It could have wide applications, and can be applied in many OPP works. Besides, redundant PMUs within the full ZIC can be shared within large areas post contingency. A concept of redundancy sharing is proposed for post-contingency ZIC model, whose observability can be guaranteed with proposed redundancy-sharing conditions. Comprehensive case studies are conducted using the proposed approaches. The simulation results for both IEEE cases and a real-world system show the proposed method can further reduce PMU deployment number.**

***Index Terms*—Optimal PMU placement, Zero-Injection Cluster, Cluster Search, Redundancy Sharing, Integer Linear Programming**

I. Introduction

Phasor measurement units (PMU) are high-precision devices capable of providing synchronized phasor measurements of voltages and currents [1], [2]. A reference time signal provided by the global positioning system (GPS) renders the synchronization of acquired data [3]. In response to the demand for system state observability in intelligent power systems, PMUs serve as the most suitable technological devices for wide-area measurement systems [4], [5]. PMU applications range from state estimation to frequency stability, and disturbance/outage monitoring [6], [7]. However, its cost and space constraints are often the barriers limiting PMU deployment [8]. Technically, having PMU placed on every node is unnecessary, as the PMU can capture the nodal voltage as well as branch currents incident to it [9], [10]. Hence, achieving the overall observability of systems via the minimum number of PMUs has become an interesting problem for PMU placement, namely the optimal PMU placement (OPP) problem [11], [12].

The OPP problem is inherently an NP complex problem with a decision space of 2^N possible solutions for an N-node system [13]. In recent years, a diversity of algorithms and methods for determining the optimal PMU placement have been proposed, which can be generally classified into two categories [13], [14], namely heuristic and deterministic methods. Researchers have employed interesting heuristic methods to optimally place PMUs, such as genetic algorithms [15], [16], particle swarm optimization [17], imperialistic competition algorithms, TABU search [18], etc. By elaborating computer-recognizable observability criteria, the feasibility and optimality of the solutions are repeatedly analyzed during the convergence process. In contrast, well-received deterministic techniques have also been applied to solve the OPP problem, including integer linear programming (ILP) [14], [19], integer nonlinear programming [20], semidefinite programming (SDP) [21], etc.

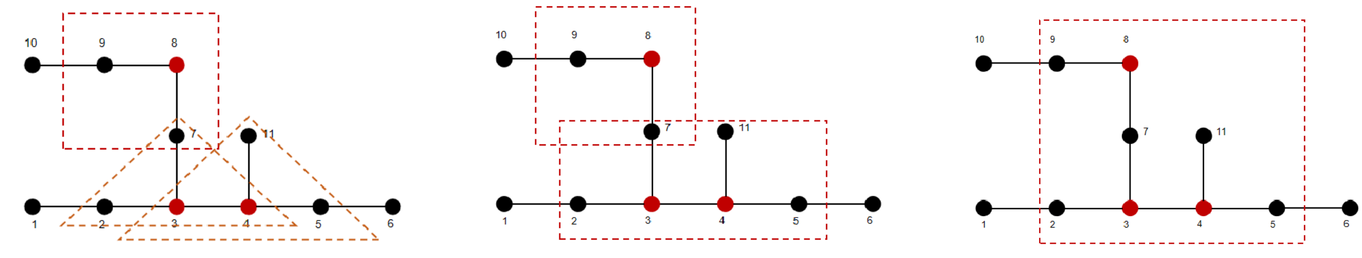
In the OPP literature, zero-injection nodes (ZINs) or zero-injection buses (ZIBs) , i.e., nodes/buses not connected with any generator or load, are widely leveraged for developing observability conditions that can provide additional information. ZINs are often named in distribution system while ZIBs are typically defined in bulk system. As the techniques discussed in this work apply for both distribution and bulk systems, we refers ZIN/ZIB as ZIN unless specified. The number of reducible PMUs is positively related to the number of ZINs. Xu et al. [22] improves the OPP model by reconfiguring constraints related to ZIN. They elaborate logical expressions by performing AND and OR operations on the basic OPP constraints. Authors in [13] model the ZIN effect by introducing auxiliary variables. All ZIN-related constraints can be dropped after recasting the model. Instead of treating each ZIN independently, models established in [23] take advantage of adjacent ZINs by hypothetically merging them into a single bus. In reference [16] and [24], numerical rules for neighboring ZINs are proposed. Their experimental results indicate that considering adjacent ZINs can further reduce the number of PMUs. Notably, due to the lack of a deterministic mathematical formulation, both methods adopted heuristic algorithms for repetitively checking whether the proposed criterion is met. Subsequently, Abdulrahman et al. [25] proposed the idea of zero-injection cluster (ZIC) to incorporate neighboring ZINs and all the incident nodes together.

Generally, existing ZIN models can be classified into two categories, i.e. the basic ZIN model and advanced model considering adjacent ZINs. The former sets separate observability constraints for each ZIN. The latter tries to consider adjacent ZINs as a whole. However, most advanced models in the literature only include directly-adjacent ZINs, missing regular nodes connected ZINs. This is referred to as partial Zero-Injection Cluster (ZIC) for differentiation. We find that covering ZINs

Corresponding author: Yinyin Ge (geyinyin@xjtu.edu.cn).

H. Ye and C. Tian are with the department of automation science and engineering, and Y. Ge is with the department of cyber science and engineering, at Xi'an Jiaotong University. (yehxing@xjtu.edu.cn; tiancy@stu.xjtu.edu.cn)

L. Wu is with the department of electrical and computer engineering at Stevens Institute of Technology (lwu11@stevens.edu).



(a) Basic ZIN model

(b) Partial-ZIC model

(c) Full-ZIC model proposed in this paper

Fig. 1: Illustration of different ZIN models. Nodes 3, 4, and 8 are ZINs. Nodes in the same dashed box are modeled together.

that connected via regular nodes can significantly reduce the PMU number in many cases. Fig. 1 gives an illustration of different ZIN models. Fig. 1 (a) depicts the basic ZIN model, in which ZINs 3, 4, and 8 are considered independently. Fig. 1 (b) shows the partial ZIC model that only considers directly-adjacent ZINs, i.e., 3 and 4, resulting two ZICs. The first ZIC includes nodes 9, 8, and 7. The second ZIC is nodes 7, 11, 2, 3, 4, and 5. Fig. 1 (c) illustrates the proposed full-ZIC model, in which there is only one ZIC due to node 7 connects ZINs 8 and 3.

In addition to reducing the total deployment cost attributed to the minimized PMU numbers, researchers also attempt to incorporate various practical contingencies into the OPP model so as to enhance PMU-based measurement systems [8], [26], [27], [28]. Two categories of contingencies, i.e., branch outage and PMU outage, are typically considered in the existing literature. For example, PMU loss is considered in [14], [29]. Single branch outages are considered in [30], and a binary search algorithm and measurement redundancy strategy are proposed to minimize the number of planned PMUs. Authors in [27] further propose joint optimal placement of PMU and flow measurements considering N-2 transmission outages. Communication limitations can be considered as well, such as [13], [28]. Researchers also investigate the topology change and network reconfiguration, such as [31]. However, most of the above literature overlooks the ZIC and redundancy sharing property, which can help further reduce the number of PMUs.

To conclude, representative OPP models in recent years have been listed in Table I. Some characteristics are summarized in the table. To bridge the gap, we propose a novel ILP-based OPP model by leveraging full ZICs and observability redundancy. The main contributions of this work are listed below:

1) A novel full-ZIC model and scalable full-ZIC search algorithm are proposed in this work. The full ZIC is defined as a group of connected nodes those are either ZINs or adjacent to ZIN. Specifically, the ZIC considers nodes connected to one or multiple ZINs. The latter scenario frequently occurs and can help reduce PMU number. However, that is often ignored in the existing literature. A question of finding clusters then naturally arises. We design a dynamic adjacency matrix based algorithm to systematically find all full ZICs, and show that it has quadratic-time complexity. It could have wide applications in many OPP works.
2) Necessary conditions for PMU redundancy sharing coupled with ZIC are proposed for post-contingency observability. We show that a PMU can substitute any PMUs within the full ZIC to solve observability equations when redundancy-sharing conditions are met. Hence, redundancy-sharing PMUs are employed to maintain observability in N-1 contingency scenarios. A major difference from the existing literature is that the redundancy within the ZIC can be shared with more nodes. The proposed redundancy-sharing approach can effectively reduce the number of redundant PMUs.

The paper is organized as follows. In Section II, we present the basic optimal PMU placement model. Section III proposes a novel full-ZIC-based OPP model and a scalable full-ZIC search algorithm. The redundancy sharing techniques are presented in Section IV. We conduct case studies in Section V. At last, Section VI concludes the paper.

II. Basic Optimal PMU Placement Model

Power system observability can be evaluated through numerical and topological assessment approaches [5]. As numerical assessment suffers from expensive matrix computation, topological approaches are commonly used for observability assessment. The power system achieves full observability if all nodes are observable. A node in the network is identified as observable if its voltage vector can be directly or indirectly measured by PMU.

Case A—Directly measured node: If a node is installed with a PMU, the nodal voltage and current vectors of lines connected to it can be directly measured by PMU. We assume the PMU has enough channels.

Case B—Indirectly measured node: Nodes adjacent to directly measured nodes are referred to as indirectly measured nodes. Given the voltage at one end of a line as well as the line current and impedance, the other nodal voltage can be attained following the Ohm's Law.

The objective of a basic OPP model is to minimize the number of PMUs to achieve full observability. The model can

TABLE I: A Summary of Some OPP Models*

Reference	Year	ZIN Effect	Observability Degree	PMU Loss	Other Focus	Solution Approach
[32]	2024	✗	✓	✓	Channel limitation	A Graph Convolutional Network-Based Deep Deterministic Policy Gradient Algorithm
[28]	2024	✗	✓	✓	Line outage; Communication bandwidth	Node-Level Data Pruning Algorithms
[33]	2024	✗	✓	✓	-	Genetic Algorithm
[34]	2023	Basic ZIN Model	✗	✗	Fixed μPMU number; SCADA measurements	Mixed-integer Semidefinite Programming
[35]	2023	✗	✗	✓	Pre-existing conventional measurements; Channel limitation	Mixed-integer Semidefinite Programming
[36]	2023	Basic ZIN Model	✓	✓	-	A Two-Archive Algorithm Based on Fuzzy Prediction
[37]	2023	Basic ZIN Model	✗	✓	Communication infrastructure; Channel limitation	Genetic Algorithm
[38]	2023	Partial-ZIC Model	✓	✓	The existence of distributed generations	Integer Linear Programming
[39]	2022	Limited ZIN Effect	✓	✗	Observability propagation depth	Mixed-Integer Linear Programming
[31]	2022	Usable Zero-Injection Phase Model	✓	✗	Unique distribution network attributes	Integer Linear Programming; Genetic Algorithm; Particle Swarm Optimization Algorithm
[40]	2021	Full-ZIC Model in Some Cases**	✗	✓	Channel limitation; Pre-existing conventional measurements; Feedback control signals	Binary Interger Linear Programming
[41]	2020	Partial-ZIC Model	✗	✓	-	Integer Linear Programming
[15]	2019	✗	✓	✗	Bus vulnerability	Elitist Genetic Algorithm
[42]	2019	✗	✗	✗	Fixed PMU number	A Greedy Algorithm
[43]	2019	✗	✓	✗	The sum of variance of the robust estimators	Monte Carlo Experiments
[25]	2018	Partial-ZIC Model	✗	✓	-	Integer Linear Programming
[44]	2018	Basic ZIN Model	✗	✗	Line outage	An Opposition-Based Elitist Binary Genetic Algorithm
[29]	2015	Partial-ZIC Model	✗	✓	Pre-existing conventional measurements	Integer Linear Programming
[13]	2010	Basic ZIN Model	✗	✓	Line outage; No PMU at zero-injection bus	Integer Linear Programming
[16]	2009	Partial-ZIC Model	✗	✗	-	Immunity Genetic Algorithm
[30]	2008	Basic ZIN Model	✗	✓	-	A Binary Search Algorithm
[19]	2008	Basic ZIN Model	✗	✗	The depth of unobservability	Integer Linear Programming
[14]	2008	Basic ZIN Model	✓	✓	Phasing of PMU placement	Integer Linear Programming
[22]	2004	Partail-ZIC Model	✗	✗	Pre-existign conventional measurements	Integer Linear Programming

* There is no report on algorithm to search ZIC or partial-ZIC in the literature.
** Observability conditions only work in some cases. One has to validate the solution post-optimization and heuristically adjusts results for large systems.

be formulated as follows:

$$(P1) \quad \min_{x} \quad \sum_{i=1}^{N} x_i \tag{1}$$

$$\text{s.t.} \quad \sum_{j} a_{ij} x_j \geq b_j, \forall i. \tag{2}$$

$$x_i \in \{0,1\}, \forall i$$

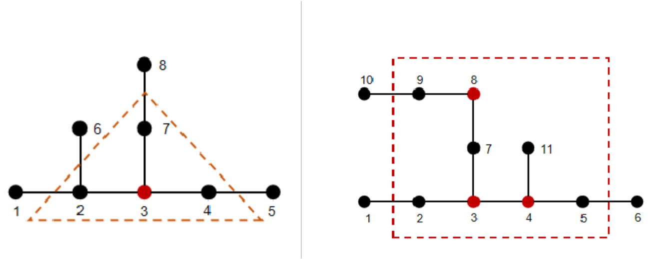
where the binary decision variable $x_i = 1$ if a PMU is installed at node i, and 0 otherwise. a_{ij} is defined as

$$a_{ij} = \begin{cases} 1 & \text{if node } i \text{ and } j \text{ are connected or } i = j \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

In fact a_{ij} is the element in the adjacent matrix. Constraint (2) denotes that at least b_j PMUs are connected to node i or directly installed at i. In other words, node i is measured via Case A or B methods. The parameter b_j can be set to 1 or another positive integer in practice.

III. Full Zero-injection Cluster

In the ZIN literature, the basic idea is to establish one equation based on Kirhhoff's Current Law (KCL) for ZIN, which often helps reduce one PMU. In contrast, ZIC-based model considers a set of ZINs together, reducing even more PMUs than ZIN. The **full** ZIC defined in this work can cover more than 50% nodes in many cases. In this section, we first present a novel full-ZIC OPP model, and then propose a scalable full-ZIC search algorithm.



(a) full ZIC with single ZIN (b) full ZIC with multiple ZINs

Fig. 2: Two illustrative examples of ZIC. (a) Node 3 is ZIN, and the ZIC includes nodes 2-4 and 7. (b) Nodes 3, 4, and 8 are ZIN, and the ZIC includes nodes 2-5, 7-9, and 11.

A. Definition and Observability of Full ZIC

The full ZIC is defined as a set of connected nodes that include at least one ZIN, and all nodes are either ZIN or adjacent to ZIN. For example, Fig. 2 presents two typical ZICs. Fig. 2 (a) has one ZIN, i.e., Node 3, and the ZIC includes nodes 2, 3, 6, and 7. Fig. 2 (b) has three ZINs, i.e., nodes 3, 4, and 8. We call nodes in ZIC **Case C** type.

Next, we show how to model the full ZIC in the OPP scheme. Assume line impedance is known. Let k be the cluster index and $\mathcal{C}_k$ be the set of all nodes in ZIC k, and $\mathcal{E}_k$ be the set of edges connecting nodes in $\mathcal{C}_k$. Let $\mathcal{I}_k$ be ZIN set in the cluster k, we have

$$V_i = V_j + Z_{ij} I_{ij}, \forall (i,j) \in \mathcal{E}_k \tag{4}$$

$$\sum_{j \in \mathcal{A}_i} I_{ij} = 0, \forall i \in \mathcal{I}_k, \tag{5}$$

where V_i is the voltage of node i, I_{ij} is the current in line ij and Z_{ij} denotes the impedance of line ij. Equation (4) stands for the voltage drop. Equation (5) follows KCL. Regarding the equation number, we establish the following proposition.

Proposition 1: For ZIC k, the total number of equations for (4) and (5) is $2|\mathcal{E}_k| + 2|\mathcal{I}_k|$, and

$$2|\mathcal{E}_k| + 2|\mathcal{I}_k| \geq 2|\mathcal{C}_k| + 2|\mathcal{I}_k| - 2, \tag{6}$$

where $|\cdot|$ denotes the number of elements in a set.

It is noted that voltage and current are complex numbers. The number of known values required for solving all the $2|\mathcal{E}_k| + 2|\mathcal{I}_k|$ state variables depends on the number of independent equations of (4) and (5).

Proposition 2: If ZIC only has one ZIN and all nodes adjacent to it are observable, then only ZIN is observable according to equations (4) and (5).

Proposition 2 is often applied in the existing ZIN literature [22], [15], [31].

Let t_i be the indicator whether the KCL equation generated from ZIN i is independent of the voltage equations, i.e.,

$$t_i = \begin{cases} 0 & i \text{ and neighboring nodes are Case A/B} \\ 1 & \text{otherwise} \end{cases} \tag{7}$$

For ZIC observability, we have following proposition.

Proposition 3: Let z_i be the indicator of whether node i is Case A or B observable, i.e.

$$z_i = \begin{cases} 1 & \text{if } \sum_{j=1}^{N} a_{ij} x_j \geq 1 \\ 0 & \text{otherwise} \end{cases}. \tag{8}$$

Then, all nodes in $\mathcal{C}_k$ are observable as long as

$$\sum_{i \in \mathcal{C}_k} z_i \geq |\mathcal{C}_k| - \sum_{j \in \mathcal{I}_k} t_j. \tag{9}$$

As there are $2|\mathcal{E}_k| + 2\sum_{i \in \mathcal{I}_k} t_i$ independent equations, we need at least $2|\mathcal{C}_k| - 2\sum_{i \in \mathcal{I}_k} t_i$ known values to solve all the equations following (4) and (5).

Case C—ZIC node: All Case C type nodes within ZIC k are observable when inequality (9) is met. Nodal voltage and line currents can be attained by solving a group of independent linear equations (4)(5). For example, we can establish six KCL equations together by leveraging three ZINs in Fig. 2 (b). In this case, Node 7 is counted only once. In contrast, Node 7 is counted twice in partial-ZIC models according to Fig. 1 (b). How it can reduce PMU number will be illustrated soon in Fig. 3.

According to Proposition 3, we formulate a new OPP model as follows:

$$\begin{aligned}
(P2) \quad \min_{x} \quad & \sum_{i=1}^{N} x_i \\
\text{s.t.} \quad & \sum_{j} a_{ij} x_j \geq 1, \ i \in \mathcal{N} \setminus \mathcal{C} && (10) \\
& \sum_{i \in \mathcal{C}_k} z_i + \sum_{j \in \mathcal{I}_k} t_j \geq |\mathcal{C}_k|, k \in \mathcal{K} && (11) \\
& M z_i \geq \sum_{j=1}^{N} a_{ij} x_i, \forall i \in \mathcal{C} && (12) \\
& z_i \leq \sum_{j=1}^{N} a_{ij} x_i, \forall i \in \mathcal{C} && (13) \\
& t_i \leq \sum_{j} a_{ij}(1 - z_j), \forall i \in \mathcal{I} && (14) \\
& M t_i \geq \sum_{j} a_{ij}(1 - z_j), \forall i \in \mathcal{I} && (15) \\
& x_i, z_i, t_i \in \{0, 1\}, \quad \forall i,
\end{aligned}$$

where $\mathcal{K}$ is the set of cluster indices, and $\mathcal{C} = \cup_{k \in \mathcal{K}} \mathcal{C}_k$, $\mathcal{I} = \cup_{k \in \mathcal{K}} \mathcal{I}_k$, and M can be set to $\max \sum_{j=1}^{N} a_{ij}$. The constraint (10) is enforced for nodes not within any ZICs, and constraint (11) from Proposition 3 is enforced for ZIC k. Compared with (P1), model (P2) often requires less PMUs to observe nodes in ZIC.

B. Full-ZIC Search Algorithm

Given the system topology and ZIN locations, a search algorithm for full ZIC is proposed in this section, as illustrated in Algorithm 1. The proposed algorithm groups nodes into full ZICs based on dynamic adjacency matrix.

It first extracts the row vectors corresponding to ZINs from A and forms a submatrix $\tilde{A}$. Each row vector of $\tilde{A}$ represents a cluster with at least one ZIN. $\tilde{A}_{ij} = 1$ means node j is within cluster i. If the sum of the jth column of $\tilde{A}$ is greater than one, node j is within multiple clusters. Nodes in these clusters can be grouped into a larger ZIC, so we conduct logical OR operation on row vectors $\{\tilde{A}_i | \tilde{A}_{ij} > 0\}$ in Line 7. Therefore, a new ZIC is attained. We then modify $\tilde{A}$ by replacing the first row vector in $\{\tilde{A}_i | \tilde{A}_{ij} > 0\}$ with the newly formed one and setting the rest vectors in $\{\tilde{A}_i | \tilde{A}_{ij} > 0\}$ as zero vectors. This process is repeated until all full ZICs are found, i.e. $\{j | \sum_i \tilde{A}_{ij} \geq 2\} = \phi$.

It can be observed that the computation burden is mainly from the first loop, i.e. line 4-13. The time complexity for this loop is no more than $\mathcal{O}(|\mathcal{I}||\mathcal{N}|)$. Therefore, the proposed algorithm has square-time complexity.

Algorithm 1 Full-ZIC Search Algorithm

Input: adjacency matrix A, ZIN set $\mathcal{I}$
Output: ZICs
1: Initialize $\tilde{A} \triangleq (a_{ij})_{|\mathcal{I}| \times N}, \forall i \in \mathcal{I}, \forall j \in [1, N]$.
2: Initialize $\mathcal{T} \triangleq \{j | \sum_{i=1}^{|\mathcal{I}|} \tilde{A}_{ij} \geq 2\}$.
3: Initialize $\mathcal{U} = \phi$.
4: **while** $\mathcal{T} \neq \phi$ **do**
5: $\quad\mathcal{U} \leftarrow \{i | \tilde{A}_{i\mathcal{T}_1} > 0\}$.
6: $\quad$**for** $j = 1 : |\mathcal{N}|$ **do**
7: $\quad\quad\tilde{A}_{\mathcal{U}_1 j} \leftarrow \bigvee_{i \in \mathcal{U}} \tilde{A}_{ij}$.
8: $\quad$**end for**
9: $\quad$**for** $i = 2 : |\mathcal{U}|$ **do**
10: $\quad\quad\tilde{A}_{\mathcal{U}_i} \leftarrow [0, 0, ..., 0]_{1 \times |\mathcal{N}|}$
11: $\quad$**end for**
12: $\quad\mathcal{T} \leftarrow \{j | \sum_{i=1}^{|\mathcal{I}|} \tilde{A}_{ij} \geq 2\}$.
13: **end while**
14: $k = 1$.
15: **for** $i = 1 : |\mathcal{I}|$ **do**
16: $\quad$**if** $\{j | \tilde{A}_{ij} > 0\} \neq \phi$ **then**
17: $\quad\quad\mathcal{C}_k \triangleq \{j | \tilde{A}_{ij} > 0\}$.
18: $\quad\quad k \leftarrow k + 1$.
19: $\quad$**end if**
20: **end for**
21: **return** $\mathcal{C}$

IV. Redundancy Sharing

Another advantage of the full ZIC is that PMU observability can be shared over a larger area. The power system security is typically evaluated by the N-1 safety criterion, which requires the system to maintain stable operation even if one component is disconnected due to contingency. To avoid failing the N-1 security rule, redundancy is often maintained in the system. Placing two PMUs at each node can guarantee full topological observability in post-contingency. However, it is not practically viable due to the PMU deployment cost or space constraint. Inspired by the N-1 contingency rule, we propose using the minimum PMUs to maintain full observability under PMU contingency.

A. Redundancy Sharing

Proposition 4: With single PMU contingency, to maintain the observability of a ZIC, the following necessary condition must be met:

$$\sum_{i \in \mathcal{C}_k} \sum_j a_{ij} x_j \geq 2|\mathcal{C}_k| - 2|\mathcal{I}_k|, \tag{16}$$

and (9) hold. It is called $n-1$ *redundancy sharing*.

Considering that only ZINs adjacent to each other share redundancy, equation (16) can also be written as follows:

$$\sum_{i \in \mathcal{C}'_k} \sum_j a_{ij} x_j \geq 2|\mathcal{C}'_k| - 2|\mathcal{I}'_k| \tag{17}$$

where C' is composed of adjacent ZINs and their neighboring nodes, and $\mathcal{I}'$ is composed of adjacent ZINs.

In the previous sections, $\sum_j a_{ij} x_j$ is used to evaluate if the node i is observable. In fact, the value of $\sum_j a_{ij} x_j$ also represents how robust the observability is. For example, the node would lose observability in N-1 post-contingency if $\sum_j a_{ij} x_j = 1$. In contrast, the node is still observable with one PMU contingency if $\sum_j a_{ij} x_j \geq 2$. Therefore, we introduce variables ϕ_i and π_k to denote the observability degrees of node i and ZIC k, respectively. The constraints are formulated as

$$\sum_j a_{ij} x_j \geq \phi_i, \; i \in \mathcal{N} \setminus \mathcal{C} \tag{18}$$

$$\sum_{i \in \mathcal{C}_k} z_i + \sum_{j \in \mathcal{I}_k} t_j \geq |\mathcal{C}_k| + \pi_k, \; k \in \mathcal{K} \tag{19}$$

where ϕ_i and π_k can be set 1. Equation (18) denotes the observability degree for regular nodes, and (19) represents the observability for the ZIC.

B. Redundancy Sharing Assisted n-1 Contingency Model

With Proposition 4, a two-step method is developed for maintaining the system's observability with one PMU contingency. In the first step, (P3) is solved for screening all solutions satisfying necessary conditions. In the second step, a necessary and sufficient solution is generated from optimal solutions to (P3).

$$\begin{aligned}
(P3) \quad & \min_x \sum_{i=1}^N x_i \\
& \text{s.t.} \sum_j a_{ij} x_j \geq 2, i \in \mathcal{N} \setminus \mathcal{C} \\
& \sum_{i \in \mathcal{C}'_k} \sum_j a_{ij} x_j \geq 2|\mathcal{C}'_k| - 2|\mathcal{I}'_k| \\
& (9)(12)(13)(14)(15) \\
& x_i \in \{0, 1, 2\}, \forall i
\end{aligned} \tag{20}$$

In model (P3), integer variable $x_i = 2$ means two PMUs are placed at node i. As a side note, the feasibility region of x_i can be changed depending on the space limit or other constraints, and the proposed technology still apply.

This two-step method is implemented by Algorithm 2. Apparently, the best solution is either included in the optimal solutions of (P3) or inferior to the optimal solutions of

(P3). Hence, Algorithm 2 first solves (P3) then verifies the optimal solutions one by one. We only check the cases when PMUs used for **Case C** nodes are lost. The post-contingency observability of nodes outside ZIC is guaranteed by equation (20). If the solution being checked guarantees the system's post-contingency observability (i.e., the sufficient condition is met), it is an optimal solution. If the solution being checked is not sufficient, extra PMUs are added till it becomes sufficient, and the modified solution is compared with the current best one in terms of PMU number. Finally, Algorithm 2 returns a necessary and sufficient solution with minimum PMUs.

Algorithm 2 OPP considering one PMU contingency

Input: adjacent matrix A, zero-injection clusters ZICs.
Output: PMU locations P_{best} with the minimum number of PMUs.

Solve (P3) and record all the optimal solutions as $P_i = \{p_{i1}, p_{i2}, ..., p_{im}\}, \forall i \in [1, k]$ where k is the number of solutions and p_{ij} represents the location of the jth PMU in P_i. .
for each $i \in [1, k]$ **do**
  Empty P_{extra}.
  for each $j \in [1, m]$ **do**
    if p_{ij} observes **Case C** nodes **then**
      $P_i' \leftarrow P_i \setminus \{p_{ij}\}$
      Check observability for PMU deployment P_i'.
      if not observable **then**
        $P_{extra} \leftarrow P_{extra} \cup \{p_{ij}\}$.
      end if
    end if
  end for
  if $i = 1$ **then**
    $P_{best} \leftarrow P_i \cup P_{extra}$.
  end if
  if $P_{extra} = \phi$ **then**
    $P_{best} \leftarrow P_i$
    Break.
  else if $|P_{extra}| + |P_i| < |P_{best}|$ **then**
    $P_{best} \leftarrow P_i \cup P_{extra}$.
  end if
end for
return P_{best}

C. Practical Applications

In the OPP problem, there are often multiple optimal solutions, which can lead to different observability degrees for normal and/or post-contingency cases. The concept of redundancy sharing can be employed to enhance the observability robustness. We present a two-stage approach as an example.

$$
\begin{aligned}
(P4) \quad \varphi = \min_{x,\phi,\pi} \quad & \sum_{i=1}^{N} x_i \\
\text{s.t.} \quad & \phi_i \geq 1,\ i \in \mathcal{N} \setminus \mathcal{C} && (21) \\
& \pi_k \geq 1,\ k \in \mathcal{K} && (22) \\
& (9), (18), (19) \\
& x_i \in \{0, 1\}, \quad \forall i.
\end{aligned}
$$

$$
\begin{aligned}
(P5) \quad \max_{x,\pi,\phi} \quad & \sum_{i \in \mathcal{N} \setminus \mathcal{C}} \phi_i + \sum_{k \in \mathcal{K}} \pi_k \\
\text{s.t.} \quad & \sum_{i=1}^{N} x_i \leq \varphi && (23) \\
& (9), (18), (19), (21), (22) \\
& x_i \in \{0, 1\}, \quad \forall i.
\end{aligned}
$$

By solving problems (P4) and (P5) sequentially, we can find the solution with maximum observability with respect to the same number of PMU deployments.

In the aforementioned parts, the objective is to minimize the PMU number for full observability. In reality, the objective function can also be modified to represent PMU cost considering channels. The full observability sometimes may not be reached with budget constraints. In these cases, we can maximize the observability degree given budget cost constraints.

V. Case Study

We conduct comprehensive simulations with different systems: 1) Distribution systems: IEEE 34-node, IEEE 37-node, IEEE 69-node and IEEE 123-node test feeders. 2) Transmission systems: IEEE 30-bus, IEEE 39-bus, IEEE 118-bus and IEEE 300-bus systems. 3) Large scale systems: 1354-bus and 2383-bus systems. 4) Real-world system: a distribution feeder in City of Suqian, China. It is assumed that the installation costs of the PMU on all nodes are the same. The simulations are conducted with Matlab and Gurobi using Intel Core (TM) i7-1165G7@2.8GHz.

A. Normal Operating Case

1) Simulation Results: Table II displays the simulation results of the proposed OPP model. The efficiency of the proposed method is verified by the fact that the optimal solutions of all IEEE systems can be obtained within 2 seconds, and it takes less than 90 seconds to solve the models for 1354 and 2383 systems.

The number of optimal solutions is also listed in Table II, which increases nonlinearly with the scale of the network. These solutions require the same number of PMUs but have different observability degrees.

2) Effectiveness analysis of full ZIC: Table III displays the distribution and composition of zero-injection clusters in different systems. We have two observations. First, a full ZIC can cover many ZINs. For example, the 6th row shows that all six ZINs are modeled into one ZIC in IEEE 30-bus system, simplifying the utilization of ZIN. Second, the full ZICs cover more than 50% nodes in most systems. As shown in the last row, 1375 out of 2383 nodes, i.e., 57.7%, are covered in the full ZICs in the 2383-bus system.

The last column "Search time" in Table III shows the time to find all ZICs by the proposed full-ZIC search algorithm. It can be observed that all full ZICs can be found within 0.004 s for systems with less than 300 nodes. The last row shows that

TABLE II: PMU Placement with full ZICs considered

IEEE Systems	No. of PMUs	PMU location (one of all optimal solutions)	Observability	Computational time (s)	No. of optimal solutions
34-node	11	802;808;820;824;834;838;840;846;854;864;890	complete	0.0328	180
37-node	9	701;705;707;709;710;711;714;727;735	complete	0.0626	36
69-node	18	6;9;14;18;22;26;28;34;38;40;44;50;53;55;57;62;64;68	complete	0.0553	$> 10^3$
123-node	32	1;5;14;15;19;23;29;31;38;42;45;47;50;52;55;58;63;65; 68;70;74;77;82;84;87;93;95;99;103;106;109;113	complete	0.0328	1
30-bus	7	1;5;10;12;18;24;27	complete	0.0789	110
39-bus sys	8	3;8;10;16;20;23;25;29	complete	0.0983	52
118-bus	27	3;12;15;17;21;25;28;35;40;43;49;53;56;62;69;72;75;77; 80;85;86;90;94;101;105;110;114	complete	0.1228	$> 10^3$
300-bus	47	3;11;15;22;26;35;39;41;42;47;51;55;58;64;79;86;88;93;98;101; 105;116;118;134;154;157;162;167;168;170;183;190;196;200; 210;211;214;224;227;238;267;268;269;275;278;294;297	complete	1.0120	1
1354-bus	153	–	complete	71.5990	$> 10^3$
2383-bus	509	–	complete	85.7313	$> 10^3$

TABLE III: Full ZICs in different systems (ZINs in the same ZIC are listed in the same square bracket.)

IEEE Systems	No. of ZINs	No. of ZICs	No. of nodes in ZICs	ZINs in each ZIC	Search time (s)
34-node	5	2	10	[812,814,850];[852,888]	0.0014
37-node	11	3	31	[702,703,704,705,706,707,708,709];[710,736];[711]	0.0006
69-node	20	9	44	[2,3,4,5,36];[15];[19]; [23,25];[30,31,32]; [45,46,47,49];[52]; [61]; [65,67]	0.0007
123-node	36	12	92	[3];[8,13,14,15,18,21,23,25,26,27];[30];[36,40,44,135];[51,151];[54,57,60]; [67,72,97,160];[78,81];[89,91,93];[100];[101,105,108,110];[152]	0.0039
30-bus	6	1	17	[6,9,22,25,27,28]	0.0021
39-bus	12	3	31	[1,2,5,6,9,10,11,13,14];[17,19];[22]	0.0008
118-bus	10	2	35	[5,9,30,37,38,63,64,68,81];[71]	0.0005
300-bus	109	20	213	–	0.0023
1354-bus	681	32	1156	–	0.4741
2383-bus	557	71	1375	–	0.5303

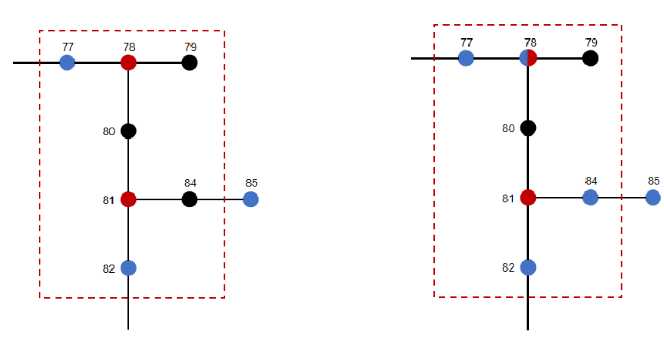
TABLE IV: Effect of full ZICs

IEEE Systems	No. of PMUs		Reduction
	W.O. ZIC	W. ZIC	
34-node	12	11	1
37-node	12	9	3
69-node	24	18	8
123-node	47	32	15
30-bus	10	7	3
39-bus	13	8	5
118-bus	31	27	4
300-bus	87	47	40
1354-bus	397	153	244
2383-bus	746	509	237

all 71 ZICs can be found in 0.5303 s for the 2383-bus system. It shows the scalability of the proposed search algorithm.

The effect of ZIC is shown in Table IV, where the numbers of PMUs with and without ZIC are compared. The comparison shows that the number of PMUs required to achieve complete observability of networks can be reduced when considering ZIC. And the advantage increases with the number of ZINs in the system. Fig.3a shows an example of how ZIC reduces the number of PMUs in the normal case scenario. Nodes 80 and 79 are observed via solving equations (24)(25) sequentially. Without ZIC, node 79 will be mathematically considered unobservable since node 80 is not directly measured by any PMU.

$$\begin{cases} V_{80} - V_{81} = I_{80,81} Z_{80,81} \\ I_{80,81} + I_{84,81} + I_{82,81} = 0 \end{cases} \tag{24}$$



(a) PMU placement in normal case

(b) PMU placement considering single PMU loss

Fig. 3: Partial simulation results of IEEE 123-node system. Nodes 78, 81 are ZIN (red dots). (a) PMUs are installed at nodes 77, 82 and 85 (blue dots). (b) PMUs are installed at nodes 77, 78, 82, 84 and 85 (blue dots).

$$\begin{cases} V_{79} - V_{78} = I_{79,78} Z_{79,78} \\ V_{80} - V_{78} = I_{80,78} Z_{80,78} \\ I_{79,78} + I_{80,78} + I_{77,78} = 0 \end{cases} \tag{25}$$

3) Comparison with Peer Works: The proposed method is compared with other works considering zero-injection nodes. Brief introductions are listed below.

- [32]: A deep reinforcement learning-based method. The ZIN effect is not considered.
- [3]: A mixed-integer linear programming model that adopts the basic ZIN model.

- [36]: A Multiobjective intelligent decision -making method that adopts the basic ZIN model.
- [38]:A graph theory based method that considers the adjacent ZINs, i.e., partial ZIC.
- [25]: An integer linear programming model that considers the adjacent ZINs, i.e., partial ZIC.
- [41]: An integer linear programming method that considers adjacent ZINs, i.e., partial ZIC.
- [29]: An integer linear programming model that considers the effect of adjacent ZINs, i.e., partial ZIC.
- [32]: A deep reinforcement learning based method that adopts the basic ZIN model.
- [40]: An integer linear programming model based on a full-ZIC model that only work in some cases.

Table V shows the comparison results. According to the simulation results, two conclusions can be drawn. First, the effect of ZIN can reduce the number of PMUs required for full observability. For instance, the number of PMUs obtained by [32] is 9 for IEEE 30-bus system, whereas only 7 PMUs are required as reported by [41] and [29]. Second, the full ZIC model proposed in this paper requires less PMUs than the partial-ZIC models. In the IEEE 118-bus system, there are no adjacent ZINs, and thus partial ZIC in the literature loses efficacy. In contrast, as shown in Column ”ZINs in each ZIC” Table III, nine out of ten ZINs in the IEEE 118-bus system can be covered in the full ZIC. Therefore, the proposed method requires 27 PMUs while all other methods require 28 PMUs.

B. Single PMU Contingency Scenario

1) Simulation Results: Table VI lists the minimum number and PMU locations for the test systems while considering single PMU contingencies. It is observed that considering PMU loss leads to an increase in the number of PMUs in comparison with the normal case. Solutions in Table VI ensure that no matter which PMU is lost, the systems are still observable.

2) Comparison with Peer Works: The results of the proposed method are compared with peer works in terms of the number of PMUs. As shown in Table VII, the proposed method requires the minimum number of PMUs for each of the systems. Compared with [45] and [13], the proposed method requires less PMUs even without redundancy sharing. Methods proposed in [29] and [41] also consider the effect of zero injection in one PMU contingency. The two models ensure the optimality of the solutions by enumerating all situations. However, the enumeration is realized by modifying the adjacency matrices, which does not allow 2 PMUs to be placed at the same node. Therefore, the simulation results of [29] and [41] have more PMUs compared with the proposed method.

3) Effectiveness Analysis of Redundancy Sharing: As shown in Table VIII, the number of PMUs required for post-contingency observability can be minimized when redundancy sharing is considered. The reason is that the effect of zero-injection is exploited in both normal case and the case of one PMU loss. Besides, the proposed method allows multiple PMUs deployed on one node. Restricting the number of PMUs on each node will lead to an increase in PMU numbers. The indices of nodes with two PMUs installed are listed twice in Table VI.

Fig. 3b shows how the redundancy is shared when PMU contingency occurs. In Fig. 3b, two additional PMUs are placed on nodes 78 and 84 so that the ZIC remains observable no matter which PMU is lost. All nodes in Fig. 3b can be directly measured by PMUs in normal conditions. When PMU loss occurs at node 82, the KCL equation in 24 can be utilized for observing node 82. When PMU loss occurs at node 78, nodes 79 and 80 can be observed by equations (24)(25), and node 78 can be directly measured by the PMU installed on node 77.

4) Efficiency Analysis of Algorithm 2: The efficiency of the proposed two-step method is analyzed in Table IX. The original search space of a system with N nodes is composed of 3^N solutions. In the first step, necessary solutions with minimum PMUs are screened out, forming the search space of the second step. It can be noticed that with the aid of Proposition 4, the search space of the heuristic process can be significantly reduced. In the second step, the necessary and sufficient solutions of IEEE 69-node and IEEE 30-bus systems are found from the optimal solutions in the first step. Thus, the optimality of these two solutions can be guaranteed. Notably, the solving process of the two systems terminated before all the solutions are tested, minimizing the searching process as much as possible. For other systems, i.e. IEEE 34-node, IEEE 37-node, IEEE 123-node, IEEE 118-bus, IEEE 39-bus, IEEE 300-bus, 1354-bus, and 2383-bus systems, the final solutions are generated by modifying the optimal solutions in the first step. Although the optimality of these solutions cannot be guaranteed, the algorithm returns the solution that is closest to the lower bound of the number of PMUs among the explored solutions. To further determine the minimum PMU number for these systems, all the optimal solutions to (P3) are explored.

The complexity of (P3) is compared with peer works as shown in Table X. For a system with N nodes, the proposed integer linear programming model requires no more than $3N$ key variables, including N logical variables determining whether the corresponding nodes need PMUs to be placed, N integer variables deciding the numbers of PMUs to be placed at the corresponding nodes and K logical variables for redundancy sharing, where K equals to the number of ZICs. The decision variables required for the proposed method and method in [29] have the same order of magnitude, which is $O(N)$. But the method proposed in [41] requires $O(N^3)$ variables. In terms of the number of constraints, the proposed model requires the least number of constraints among these methods. The order of magnitude of our method and other methods are $O(N)$ and $O(N^2)$, respectively. On the one hand, models with large amounts of variables and constraints may take a long time to solve and a large storage space. On the other hand, numerical problems are unavoidable for large-scale models, especially for non-convex ones. According to our experience, for large-scale systems like 1354 and 2383 -bus systems, the optimal solutions are sensitive to the parameter settings of the solver.

The computational time of the second step can be found

TABLE V: Comparison of numbers of PMUs between different methods in normal case

Method	ZIN Model	IEEE Systems									
		34-node	37-node	69-node	123-node	30-bus	39-bus	118-bus	300-bus	1354-bus	2383-bus
[32]	-	-	-	-	-	9	-	32	-	-	-
[36]	Basic ZIN Model	-	-	-	-	-	8	28	-	-	593
[3]	Basic ZIN Model	-	-	-	-	7	-	28	-	-	-
[38]	Partial-ZIC Model	11	-	18	32	-	-	-	-	-	-
[25]	Partial-ZIC Model	-	-	-	-	7	8	28	-	-	-
[41]	Partial-ZIC Model	11	10	18	34	7	8	28	53	196	535
[29]	Partial-ZIC Model	11	10	18	34	7	8	28	53	196	535
[32]	Partial-ZIC Model	-	-	-	-	-	9	32	-	-	562
[40]	Full-ZIC Model in Some Cases	-	-	-	-	7	8	28	-	-	-
Proposed	Full-ZIC Model	**11**	**9**	**18**	**32**	**7**	**8**	**27**	**47**	**153**	**509**

TABLE VI: PMU placement considering single PMU loss

IEEE Systems	No. of PMUs	PMU location	Post contingency observability
34-node	22	801;802;808;810;818;820;822;824;826;828;832;834;836;838,840;844;846;848;854;856;828;862;864;888	complete
37-node	20	701;705;706;707;707;709;710;710;711;714;718;720;728;729;730;734;737;740;742;744;799	complete
69-node	36	3;6;7;9;11;13;14;16;18;19;21;22;24;26;27;32;34;35;38;39;40;41;42;44;48;49;51;53;54;55;56;57;58;60;62;63;65;68;69	complete
123-node	64	1;2;5;6;8;9;14;15;19;20;28;30;31;32;34;36;38;39;40;42;43;45;46;47;48;50;52;55;56;58;59;62;63;65;66;68;70;71;74;75;76;77;78;82;83;84;85;87;88;91;95;96;98;100;103;104;106;107;108;109;111;113;114;116	complete
30-bus	14	2;3;4;7;10;12;13;15;17;19;20;24;27;27	complete
39-bus	17	2;5;6;7;8;16;17;20;21;23;25;26;29;34;36;37;38	complete
118-bus	55	2;3;6;8;9;11;12;15;17;19;20;21;23;27;28;29;32;34;35;40;41;43;45;48;49;50;51;52;53;56;59;62;66;70;71;71;75;77;80;84;85;86;87;89;90;92;94;96;100;101;105;106;109;110;111;112;115;117;118	complete
300-bus	115	3;11;15;19;22;23;25;35;42;47;51;55;57;60;64;79;86;88;92;93;98;101;103;105;113;116;118;119;122;132;133;134;138;141;152;157;160;167;170;172;175;177;179;183;194;200;209;210;211;214;222;224;228;237;267;268;269;274;276;277;294;299;3;11;15;19;35;42;47;51;55;60;64;79;86;88;92;93;98;113;116;133;152;157;170;177;194;200;210;211;214;237;268;269;274;294;19;51;57;60;79;92;101;103;105;113;119;132;138;141;179;194;224;267;276	complete
1354-bus	494	–	complete
2383-bus	1601	–	complete

TABLE VII: Comparison of the proposed method and other methods considering single PMU loss

Methods	IEEE Systems						
	34-node	37-node	69-node	123-node	30-bus	39-bus	118-bus
[29]	24	21	37	75	14	17	61
[45]	-	-	-	-	16	-	64
[13]	-	-	-	-	15	18	63
[41]	24	21	37	75	14	17	61
[35]	-	-	-	-	21	-	66
[38]	-	-	44	66	-	-	-
[40]	-	-	-	-	14	17	61
Proposed	**22**	**20**	**36**	**64**	**14**	**17**	**55**

TABLE VIII: Effect of redundancy sharing

IEEE Systems	No. of PMUs		Reduction
	W.O. redundancy sharing	W. redundancy sharing	
34-node	26	22	4
37-node	24	20	4
69-node	48	36	12
123-node	86	64	22
30-bus	14	14	0
39-bus	21	17	4
118-bus	62	55	7
300-bus	127	115	12
1354-bus	713	494	208
2383-bus	1783	1601	182

in Table IX. Factors affecting the computational time include the number of explored solutions and the scale of the system. For systems with less than 10 solutions to be explored, i.e., IEEE 69-node, IEEE 123-node, IEEE 30-bus, and IEEE 39-bus systems, the heuristic process terminates within 1 second. Among the systems with many solutions to be tested, smaller systems, i.e. IEEE 34-node and IEEE 37-node systems, have simulation times shorter than 10 seconds, while larger systems, i.e. IEEE 300-bus, 1354-bus, and 2383-bus systems, require computing times longer than 2000 seconds.

C. A Real-world Case

We apply the proposed techniques for a real-world PMU deployment project in Suqian, Jiangsu Province, China. The system one-line diagram is shown in Figure 4. To achieve the complete observability, at least eight PMUs should be deployed at proper locations. However, PMU deployment number is limited to three due to the budget and space constraints. Therefore, we establish a two-stage approach to maximizing the observability degree with the number constraints.

In the first stage, the number of Cases A and B nodes is maximized with the PMU number limit constraint. Together with constraints (9) (12)(13)(14)(15), the OPP program finds at most 11 Cases A and B nodes. In the meantime, we find

TABLE IX: Efficiency analysis of redundancy sharing

IEEE Systems	original search space	No. of solutions in stage 1	No. of explored solutions in stage 2	computational time in stage 2 (s)
34-node	3^N	1386	1386	0.0090
37-node		186	186	0.0244
69-node		61282	3	0.0673
123-node		1	1	0.2908
30-bus		2134	3	0.0057
39-bus		9	9	0.0122
300-bus		228	228	1.4280
1354-bus		20	20	2355.2361
2383-bus		27	27	8729.6659

TABLE X: Comparison of complexity between the proposed method and other methods

method	No. of decision variables	No. of constraints
proposed	$O(N)$	$O(N)$
[41]	$O(N^3)$	$O(N^2)$
[29]	$O(N)$	$O(N^2)$

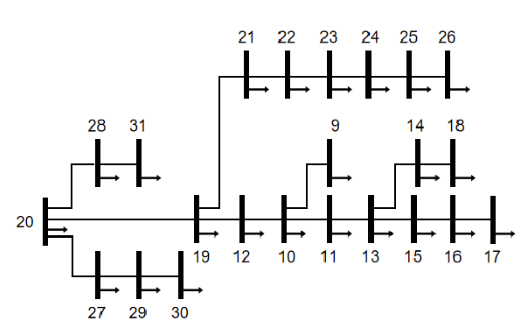


Fig. 4: A Real-world Distribution Feeder in Suqian, China

there are 16 solutions that have the same objective value. We thus formulate the second stage problem by maximizing the observability degree with 11 Cases A and B nodes constraints. The maximum observability degree is 12. Three PMUs are deployed at nodes 10, 13, and 20. The 11 observable nodes include 9-15, 19, 20, 27, and 28.

VI. Conclusion

In this paper, a concept of full ZIC is proposed for leveraging neighboring zero-injection nodes, which integrates the phenomena of direct and indirect adjacency of ZINs, as well as the phenomenon of multiple ZIN adjacency. A scalable algorithm to search all full ZICs is proposed. This paper also introduces the necessary conditions for PMU redundancy sharing in ZIC. Based on that, a two-step method is designed to maintain full observability in N-1 contingency scenarios. Notably, the established objective functions and constraints are modeled linearly, so that the OPP models can be solved using ILP solver.

Comprehensive simulations are conducted with the proposed method under both normal and contingency case scenarios. The case study shows that utilization of full ZIC can reduce the PMU number for most test systems. The redundancy-sharing method helps reduce the number of PMUs needed for maintaining complete observability in N-1 scenarios. As compared with peer works in the literature, the proposed methods achieve the minimum number of PMUs. In addition, the successful application of the proposed method on a real-world distribution feeder in China verifies its practicality. Network topology and line impedance are needed for the application of proposed techniques. One can also tailor the cost coefficients when considering the PMU channels. In conclusion, this paper provides a basic tool, which helps reduce PMU number. The techniques could have wide applications in many OPP approaches. In future study, authors plan to apply them in more complicated OPP models.

References

[1] N. M. Manousakis and G. N. Korres, "A weighted least squares algorithm for optimal pmu placement," *IEEE Trans. Power Syst.*, vol. 28, no. 3, pp. 3499–3500, 2013.

[2] A. Mishra and R. A. de Callafon, "Algebraic approach to pmu placement for minimum variance linear state estimation in power networks," *IEEE Trans. Power Syst.*, vol. 38, no. 5, pp. 4381–4390, 2023.

[3] X. Zhu, M. H. F. Wen, V. O. K. Li, and K.-C. Leung, "Optimal pmu-communication link placement for smart grid wide-area measurement systems," *IEEE Trans. Smart Grid*, vol. 10, no. 4, pp. 4446–4456, Jul. 2019.

[4] S. Azizi, G. B. Gharehpetian, and A. S. Dobakhshari, "Optimal integration of phasor measurement units in power systems considering conventional measurements," *IEEE Trans. Smart Grid.*, vol. 4, no. 2, pp. 1113–1121, 2013.

[5] R. Sodhi, S. Srivastava, and S. Singh, "Optimal pmu placement method for complete topological and numerical observability of power system," *Electr Pow Syst Res*, vol. 80, no. 9, pp. 1154–1159, 2010.

[6] A. von Meier, E. Stewart, A. McEachern, M. Andersen, and L. Mehrmanesh, "Precision Micro-Synchrophasors for Distribution Systems: A Summary of Applications," *IEEE Trans. Smart Grid.*, vol. 8, no. 6, pp. 2926–2936, Nov. 2017.

[7] M. H. F. Wen, J. Xu, and V. O. K. Li, "Optimal multistage pmu placement for wide-area monitoring," *IEEE Trans. Power Syst.*, vol. 28, no. 4, pp. 4134–4143, 2013.

[8] H. Wang, W. Zhang, and Y. Liu, "A Robust Measurement Placement Method for Active Distribution System State Estimation Considering Network Reconfiguration," *IEEE Trans. Smart Grid.*, vol. 9, no. 3, pp. 2108–2117, May 2018.

[9] B. M. B. Baldwin T. L., Mili L., "Power system observability with minimal phasor measurement placement." *IEEE Trans. Power Syst.*, no. 2, pp. 707–715, 1993.

[10] L. Huang, Y. Sun, J. Xu, W. Gao, J. Zhang, and Z. Wu, "Optimal pmu placement considering controlled islanding of power system," *IEEE Trans. Power Syst.*, vol. 29, no. 2, pp. 742–755, 2014.

[11] N. M. Manousakis, G. N. Korres, and P. S. Georgilakis, "Taxonomy of pmu placement methodologies," *IEEE Trans. Power Syst.*, vol. 27, no. 2, pp. 1070–1077, 2012.

[12] E. Castillo, A. J. Conejo, R. E. Pruneda, C. Solares, and J. M. Menendez, "$m-k$ robust observability in state estimation," *IEEE Trans. Power Syst.*, vol. 23, no. 2, pp. 296–305, 2008.

[13] F. Aminifar, A. Khodaei, M. Fotuhi-Firuzabad, and M. Shahidehpour, "Contingency-constrained pmu placement in power networks," *IEEE Trans. Power Syst.*, vol. 25, no. 1, pp. 516–523, 2010.

[14] D. Dua, S. Dambhare, R. K. Gajbhiye, and S. A. Soman, "Optimal multistage scheduling of pmu placement: An ILP approach," *IEEE Trans. Power Syst.*, vol. 23, no. 4, pp. 1812–1820, 2008.

[15] S. Almasabi and J. Mitra, "A fault-tolerance based approach to optimal pmu placement," *IEEE Transactions on Smart Grid*, vol. 10, no. 6, pp. 6070–6079, 2019.

[16] F. Aminifar, C. Lucas, A. Khodaei, and M. Fotuhi-Firuzabad, "Optimal placement of phasor measurement units using immunity genetic algorithm," *IEEE Trans. Power Syst.*, vol. 24, no. 3, pp. 1014–1020, 2009.

[17] A. Ahmadi, Y. Alinejad-Beromi, and M. Moradi, "Optimal pmu placement for power system observability using binary particle swarm optimization and considering measurement redundancy," *Expert Syst. Appl*, vol. 38, no. 6, pp. 7263–7269, 2011.

[18] H. W. Jiangnan Peng, Yuanzhang Sun, "Optimal pmu placement for full network observability using tabu search algorithm," *Int. J. Elec Power.*, vol. 28, no. 4, pp. 223–231, 2006.

[19] B. Gou, "Optimal placement of pmus by integer linear programming," *IEEE Trans. Power Syst.*, vol. 23, no. 3, pp. 1525–1526, 2008.

[20] B. Gou, "Generalized integer linear programming formulation for optimal pmu placement," *IEEE Trans. Power Syst.*, vol. 23, no. 3, pp. 1099–1104, 2008.

[21] N. M. Manousakis and G. N. Korres, "Optimal pmu placement for numerical observability considering fixed channel capacity—a semidefinite programming approach," *IEEE Trans. Power Syst.*, vol. 31, no. 4, pp. 3328–3329, 2016.

[22] B. Xu and A. Abur, "Observability analysis and measurement placement for systems with PMUs," in *IEEE PES Power Systems Conference and Exposition, 2004.*, vol. 2, New York, NY, Oct. 2004, pp. 943–946.

[23] N. H. Abbasy and H. M. Ismail, "A unified approach for the optimal pmu location for power system state estimation," *IEEE Trans. Power Syst.*, no. 2, pp. 806–813, 2009.

[24] M. Hajian, A. M. Ranjbar, T. Amraee, and A. R. Shirani, "Optimal placement of phasor measurement units: Particle swarm optimization approach," in *International conference on intelligent systems applications to power systems*, 2007, pp. 1–6.

[25] R. G. Abdulrahman I., "ILP-Based optimal pmu placement with the inclusion of the effect of a group of zero-injection buses." *J Control Autom Electr Syst*, no. 29, pp. 512–524, 2018.

[26] J. Liu, F. Ponci, A. Monti, C. Muscas, P. A. Pegoraro, and S. Sulis, "Optimal Meter Placement for Robust Measurement Systems in Active Distribution Grids," *IEEE Trans. Instrum. Meas.*, vol. 63, no. 5, pp. 1096–1105, May 2014.

[27] S. Zhu, L. Wu, S. Mousavian, and J. H. Roh, "An optimal joint placement of pmus and flow measurements for ensuring power system observability under n-2 transmission contingencies," *Int. J. Elec Power.*, vol. 95, pp. 254–265, 2018.

[28] A. K. Mandal and S. De, "Joint optimal pmu placement and data pruning for resource efficient smart grid monitoring," *IEEE Transactions on Power Systems*, vol. 39, no. 3, pp. 5382–5392, 2024.

[29] K. Gharani Khajeh, E. Bashar, A. Mahboub Rad, and G. B. Gharehpetian, "Integrated model considering effects of zero injection buses and conventional measurements on optimal pmu placement," *IEEE Transactions on Smart Grid*, vol. 8, no. 2, pp. 1006–1013, 2017.

[30] S. Chakrabarti and E. Kyriakides, "Optimal placement of phasor measurement units for power system observability," *IEEE Trans. Power Syst.*, vol. 23, no. 3, pp. 1433–1440, 2008.

[31] M. P. Anguswamy, M. Datta, L. Meegahapola, and A. Vahidnia, "Optimal micro-pmu placement in distribution networks considering usable zero-injection phase strings," *IEEE Transactions on Smart Grid*, vol. 13, no. 5, pp. 3662–3675, 2022.

[32] X. Zhou, Y. Wang, Y. Shi, Q. Jiang, C. Zhou, and Z. Zheng, "Deep reinforcement learning-based optimal pmu placement considering the degree of power system observability," *IEEE Trans. Ind Inform*, vol. 20, no. 6, pp. 8949–8960, 2024.

[33] J. Ge, Y. Xu, Z. Wu, L. Mili, S. Lu, Q. Hu, and W. Gu, "Data-driven optimal pmu placement for power system nonlinear dynamics using koopman approach," *IEEE Trans. Ind Inform*, pp. 1–12, 2024, early Access.

[34] Y. Peng, Z. Wu, W. Gu, S. Zhou, and P. Liu, "Optimal micro-pmu placement for improving state estimation accuracy via mixed-integer semidefinite programming," *Journal of Modern Power Systems and Clean Energy*, vol. 11, no. 2, pp. 468–478, 2023.

[35] T. Chen, H. Ren, Y. Sun, M. Kraft, and G. A. J. Amaratunga, "Optimal placement of phasor measurement unit in smart grids considering multiple constraints," *Journal of Modern Power Systems and Clean Energy*, vol. 11, no. 2, pp. 479–488, 2023.

[36] B. Cao, Y. Yan, Y. Wang, X. Liu, J. C.-W. Lin, A. K. Sangaiah, and Z. Lv, "A multiobjective intelligent decision-making method for multistage placement of pmu in power grid enterprises," *IEEE Trans. Ind Inform*, vol. 19, no. 6, pp. 7636–7644, 2023.

[37] R. Bhattacharjee and A. De, "A novel bus-ranking-algorithm-based heuristic optimization scheme for pmu placement," *IEEE Trans. Ind Inform*, vol. 19, no. 9, pp. 9921–9932, 2023.

[38] R. B. Mukherjee M., "Optimal μpmu placement in radial distribution networks using novel zero injection bus modelling." *SN COMPUT. SCI.*, vol. 4, no. 575, 2023.

[39] X.-C. Guo, C.-S. Liao, and C.-C. Chu, "Probabilistic optimal pmu placements under limited observability propagations," *IEEE Systems Journal*, vol. 16, no. 1, pp. 767–776, 2022.

[40] M. Elimam, Y. J. Isbeih, M. S. E. Moursi, K. Elbassioni, and K. H. A. Hosani, "Novel optimal pmu placement approach based on the network parameters for enhanced system observability and wide area damping control capability," *IEEE Trans. Power Syst*, vol. 36, no. 6, pp. 5345–5358, 2021.

[41] X. Chen, F. Wei, S. Cao, C. B. Soh, and K. J. Tseng, "Pmu placement for measurement redundancy distribution considering zero injection bus and contingencies," *IEEE Syst. J.*, vol. 14, no. 4, pp. 5396–5406, 2020.

[42] W. Li, D. Deka, M. Chertkov, and M. Wang, "Real-time faulted line localization and pmu placement in power systems through convolutional neural networks," *IEEE Trans. Power Syst*, vol. 34, no. 6, pp. 4640–4651, 2019.

[43] L. Sun, T. Chen, X. Chen, W. K. Ho, K.-V. Ling, K.-J. Tseng, and G. A. J. Amaratunga, "Optimum placement of phasor measurement units in power systems," *IEEE Trans. Instru. Meas*, vol. 68, no. 2, pp. 421–429, 2019.

[44] S. Almasabi and J. Mitra, "Multistage optimal pmu placement considering substation infrastructure," *IEEE Trans. Ind Appl*, vol. 54, no. 6, pp. 6519–6528, 2018.

[45] M. Beg Mohammadi, R.-A. Hooshmand, and F. Haghighatdar Fesharaki, "A new approach for optimal placement of pmus and their required communication infrastructure in order to minimize the cost of the wams," *IEEE Trans. Smart Grid.*, vol. 7, no. 1, pp. 84–93, 2016.